Gravity
An Introduction

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Lecture One

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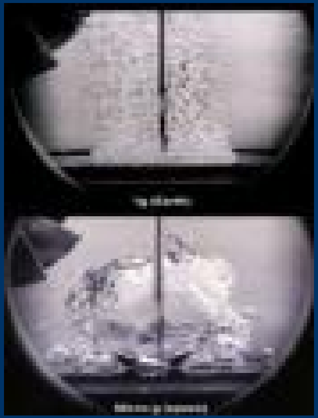
gravitation and space science

micro-g environment:

e.g. boiling water



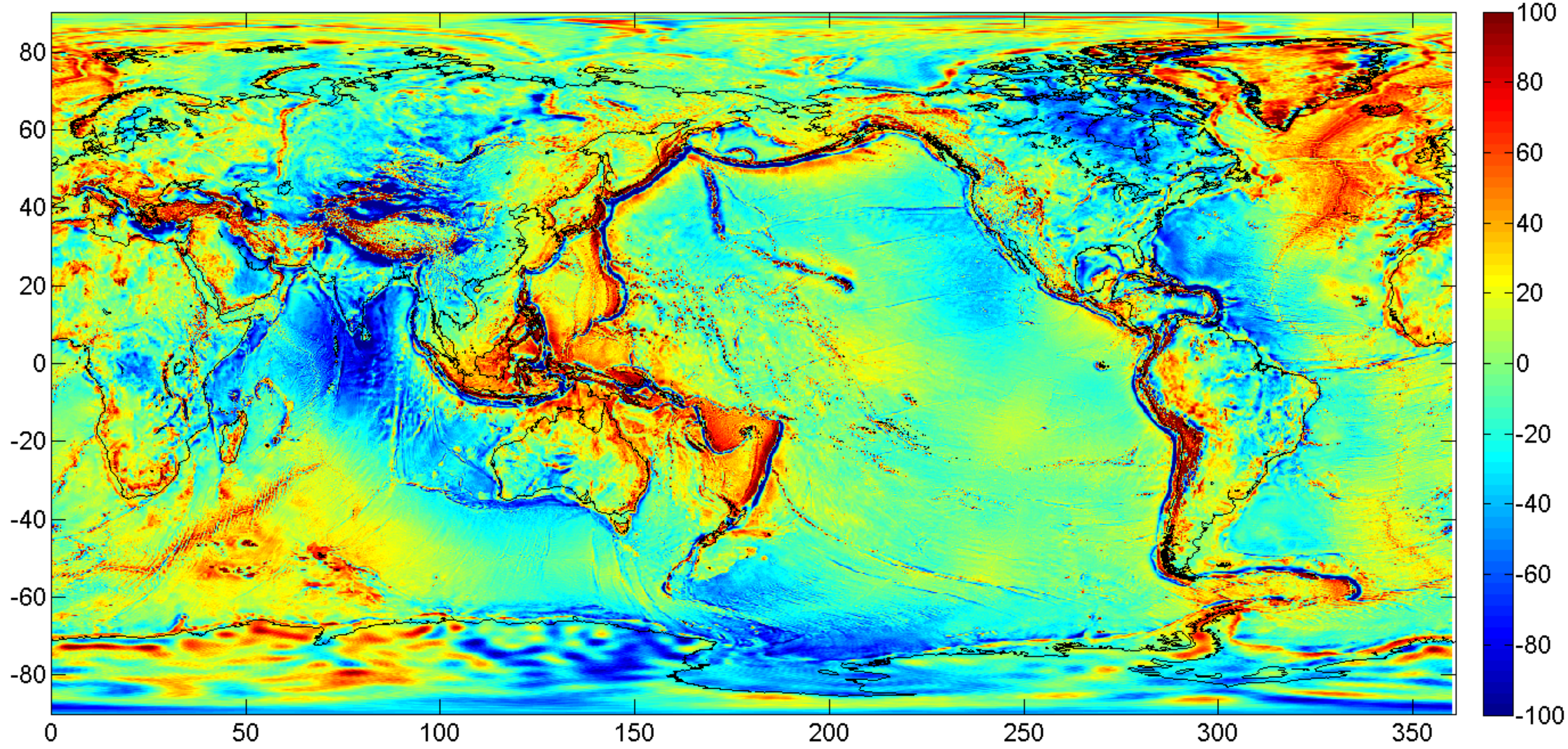
fundamental physics:

e.g. equivalence principle

earth sciences

e.g. gravitational field

of earth moon and planets



global map of gravity anomalies

the plan

Lecture One:

Theoretical basics of gravitation as applied to the earth

[gravitational law, properties, mathematical representation]

The „language“ for the two other lectures

Lecture Two:

The role of the earth‘s gravitational field in earth sciences

[gravity anomalies, geoid as a reference, temporal variations]

Lecture Three:

Principles of satellite gravimetry; in their logic derived from free fall tests

in a laboratory on earth

[the orbit, principles of GRACE, ESA‘s mission GOCE and satellite gradiometry]

introduction to gravitation

Newton's law of gravitation:

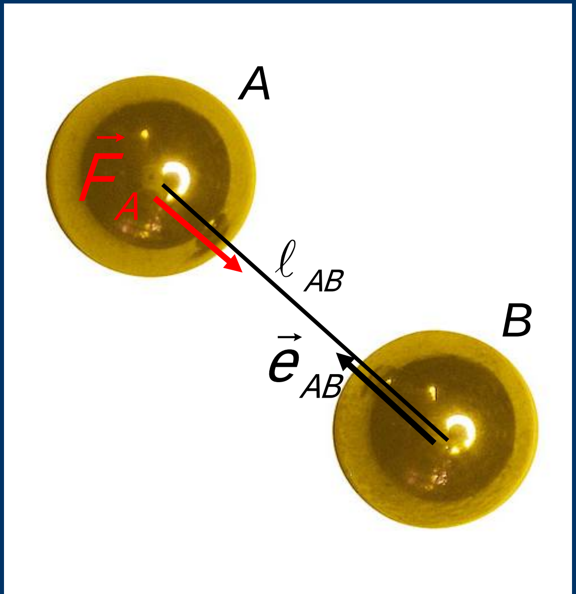
$$\vec{F}_A = -G \frac{m_A m_B}{\ell_{AB}^2} \vec{e}_{AB} = -G \frac{m_A m_B}{\left|\vec{x}_A - \vec{x}_B\right|^3} \left(\vec{x}_A - \vec{x}_B\right)$$

Newton's second law:

$$\vec{F}_A = m_A^{\prime} \; \vec{a}_A$$

gravitational acceleration:

$$\vec{a}_A = -G \frac{m_A m_B}{m_A^{\prime} \ell_{AB}^2} \vec{e}_{AB}$$



introduction to gravitation

from single mass, to many masses, to a continuum

$$\vec{a}_A = -G \iiint_{\Sigma} \frac{\rho_B}{\ell_{AB}^2} \vec{e}_{AB} \, d\Sigma_B$$

Fundamental properties of Newton‘s law of gravitation:

- central force
- action = reaction
- inverse square distance
- superposition of all partial forces
- instantaneous

introduction to gravitation

$$\vec{a}_A = -G \iiint_{\Sigma} \frac{\rho_B}{\ell_{AB}^2} \vec{e}_{AB} \, d\Sigma_B$$

$\vec{a}_A$ is a vector field in space

with the following properties:

$\nabla \times \vec{a}_A = 0$ *curl free* $\vec{a}_A = \nabla_A V$

i.e. there exists a gravitational potential V

and in outer space, we get:

$\nabla \cdot \vec{a}_A = 0$ *source free* $\nabla^2 V = 0$

$$\vec{a}_A = -G \iiint_\Sigma \frac{\rho_B}{\ell^2_{AB}} \vec{e}_{AB} \, d\Sigma_B$$

$\vec{a}_A$ is a vector field

with the following properties:

$$\nabla \times \vec{a}_A = 0 \qquad \textit{curl free} \qquad \vec{a}_A = \nabla_A V$$

i.e. there exists a gravitational potential V

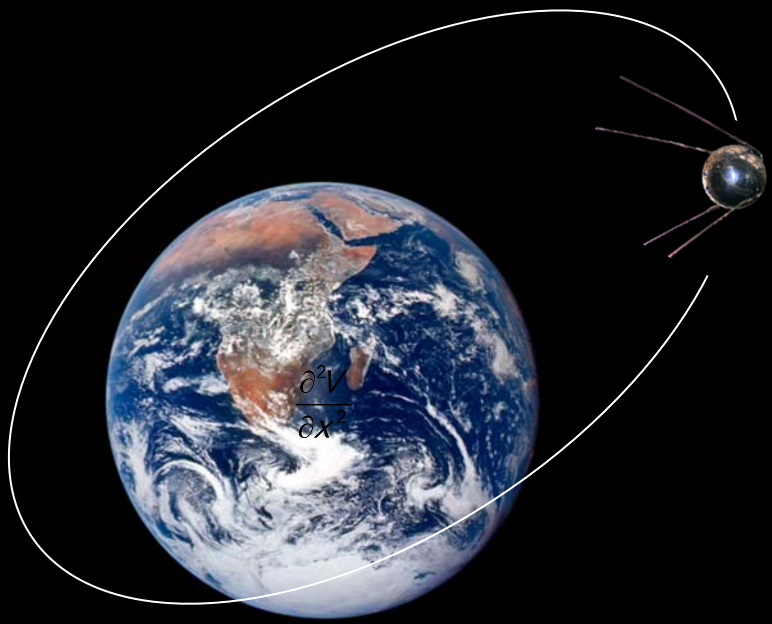
and in outer space, we get:

$$\nabla \cdot \vec{a}_A = 0 \qquad \textit{source free} \qquad \nabla^2 V = 0$$

$$V_A = G \iiint_\Sigma \frac{\rho_B}{\ell_{AB}} d\Sigma_B$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

example: satellite orbit



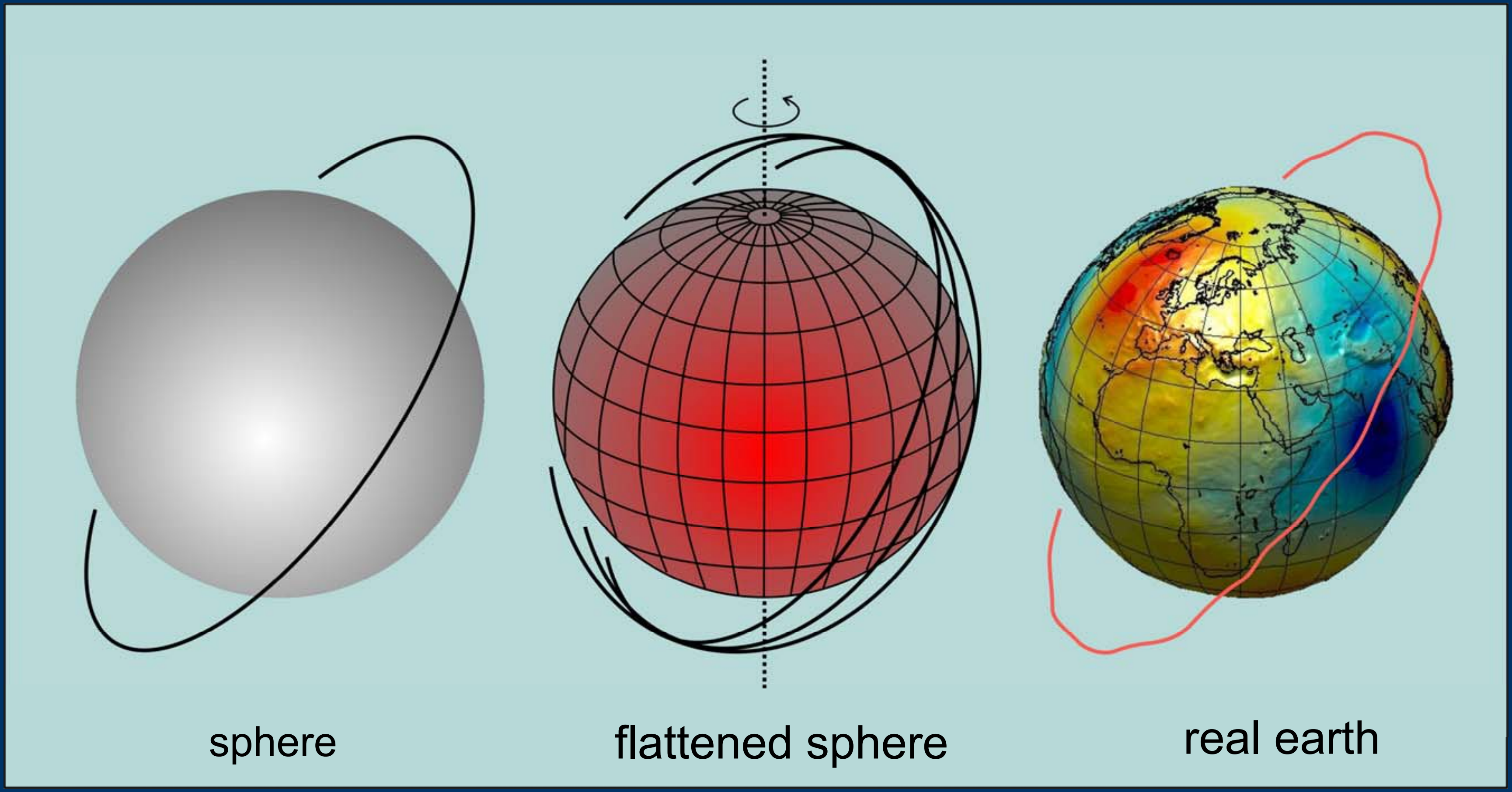
a nice application: a satellite orbit

$$\ddot{\vec{x}}_A = \vec{a}_A = \nabla_A V + \text{" } perturbations \text{ "}$$

$$and\ \ initial\ \ conditions : \vec{x}_0\ ;\ \dot{\vec{x}}_0$$

example: satellite orbit

a nice application: a satellite orbit



Kelplerian ellipse

precessing ellipse

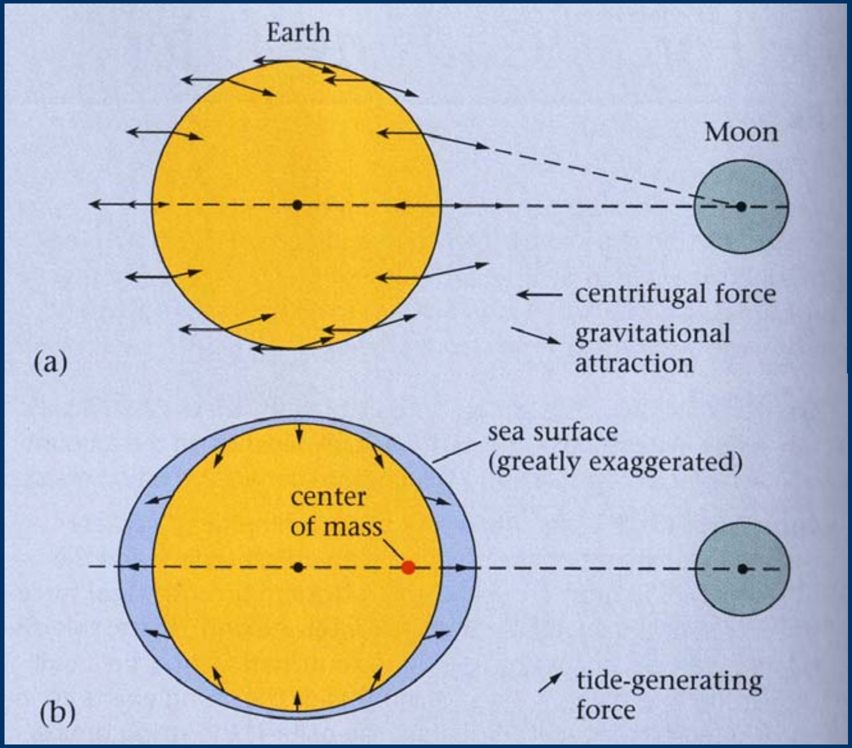
precessing ellipse
plus „gravitational code“

example: satellite orbit

What about the attraction of sun, moon and planets?

Answer: They determine the earth‘s orbit about the sun

Tides are acceleration relative to the earth‘s center of mass

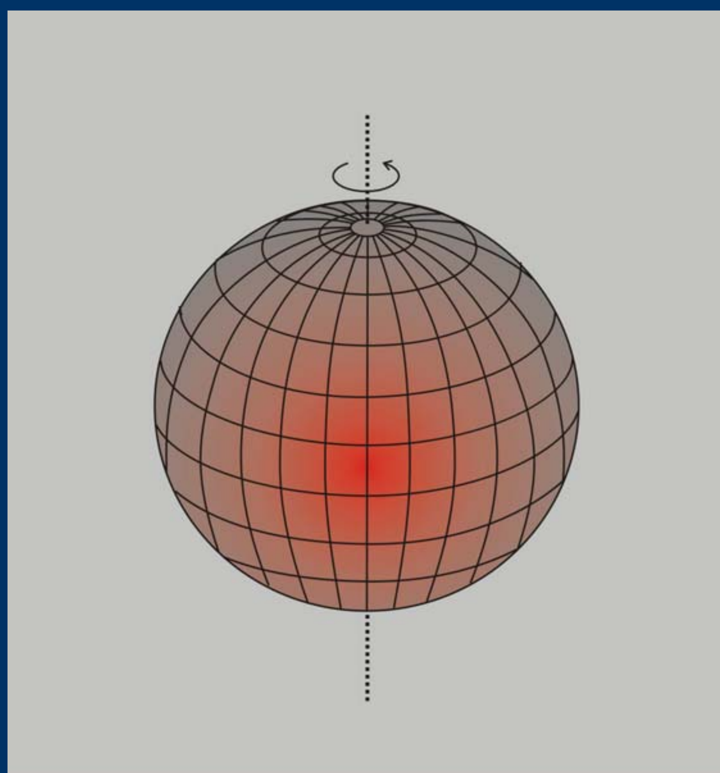


Marshak, 2005

gravitation and gravity

on the surface of the rotating earth one measures:

gravity= gravitation + centrifugal acceleration



$$\vec{g} = \vec{a} + \vec{z}$$

and

$$W = V + Z$$

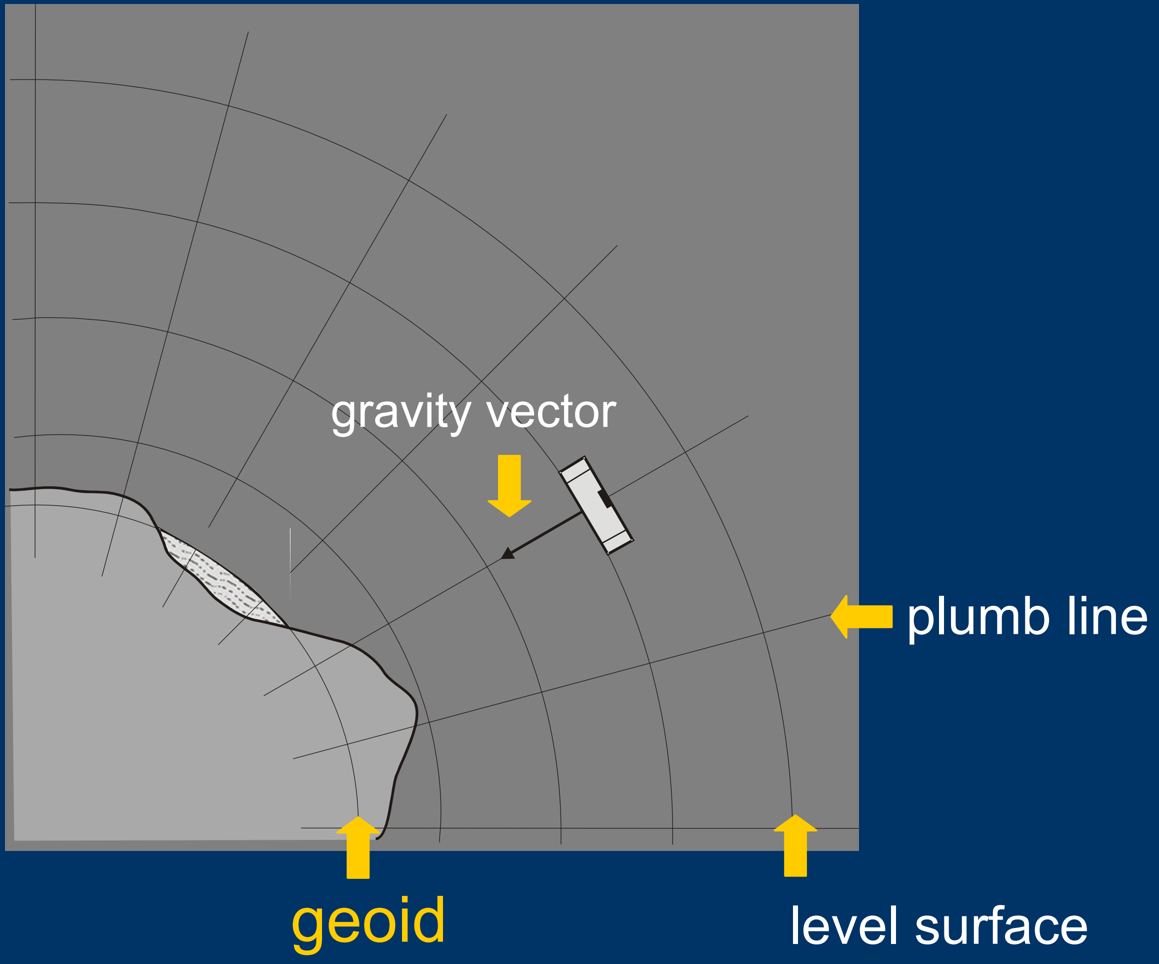
size of gravity signals

gravity (in laboratory at TU München)
9.807 246 72 m/s^2

stationary / variable

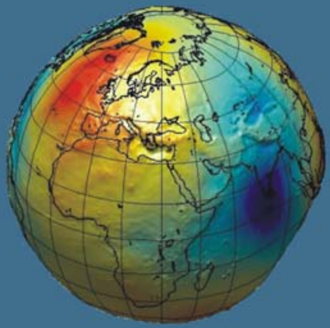
10^{0}	spherical Earth
10^{-3}	flattening & centrifugal acceleration
10^{-4}	mountains, valleys, ocean ridges, subduction
10^{-5}	density variations in crust and mantle
10^{-6}	salt domes, sediment basins, ores
10^{-7}	tides, atmospheric pressure
10^{-8}	temporal variations: oceans, hydrology
10^{-9}	ocean topography, polar motion
10^{-10}	general relativity

geometry of the earth‘s gravity field



$$W = W_0 = const.$$

gravity related quantities



geoid heights

deviation of geoid from Earth ellipsoid
or equilibrium figure

± 30m

topography

altitude of land surface above geoid (mean sea level)

< 8000m

ocean topography

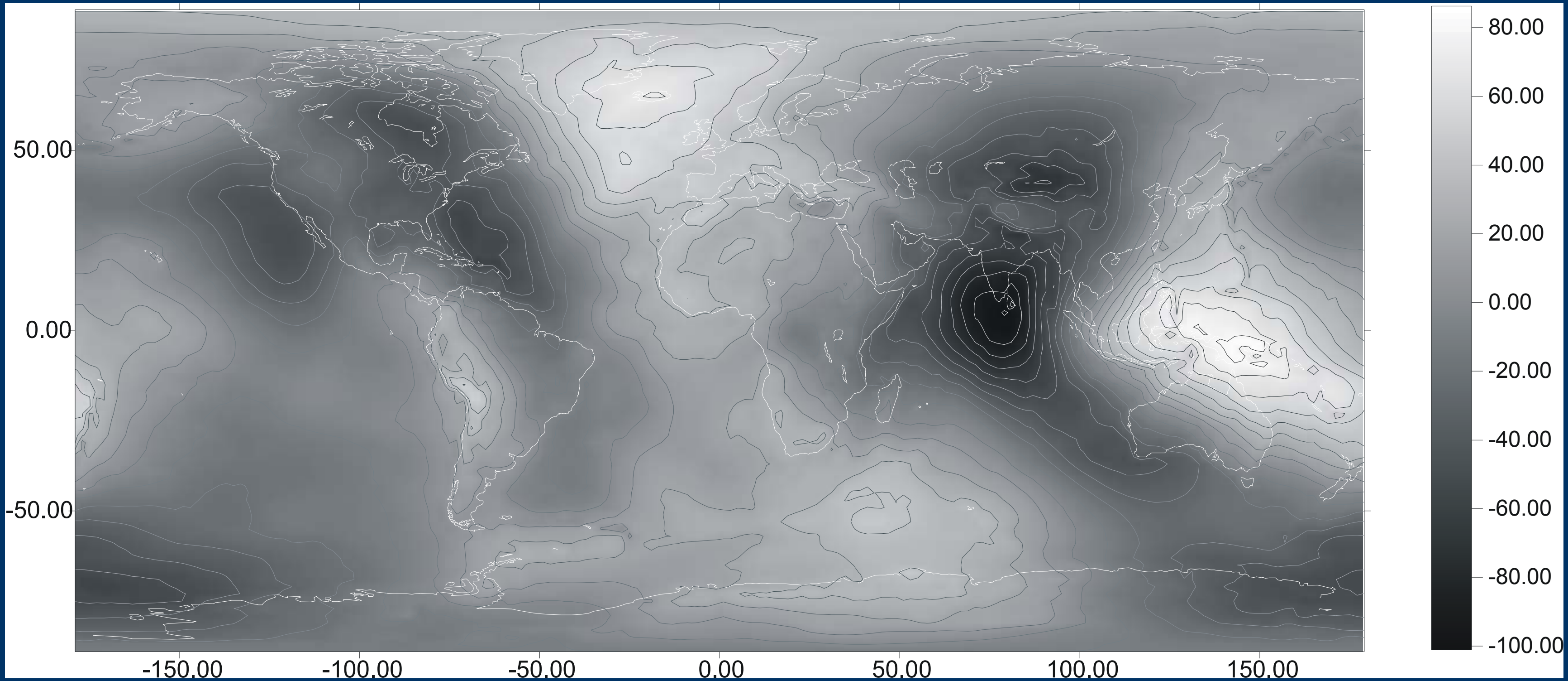
deviation of actual ocean surface from
geoid (= idealised ocean surface)

<1m to 2m

gravity related quantities

map with geoid heights

relative to the GRS80 ellipsoid



series representation of gravitational field

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

Laplace equation (PDE)

solution in Cartesian coordinates (after determination):

$$\delta V_A(x, y, z) = V_0 \sum_{k=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} \exp\left(-\sqrt{k^2 + \ell^2}\, z\right) c_{k\ell} \exp\left(i\left[kx + \ell y\right]\right)$$

$$\text{with} \quad z \geq 0$$

solution in spherical coordinates (after determination):

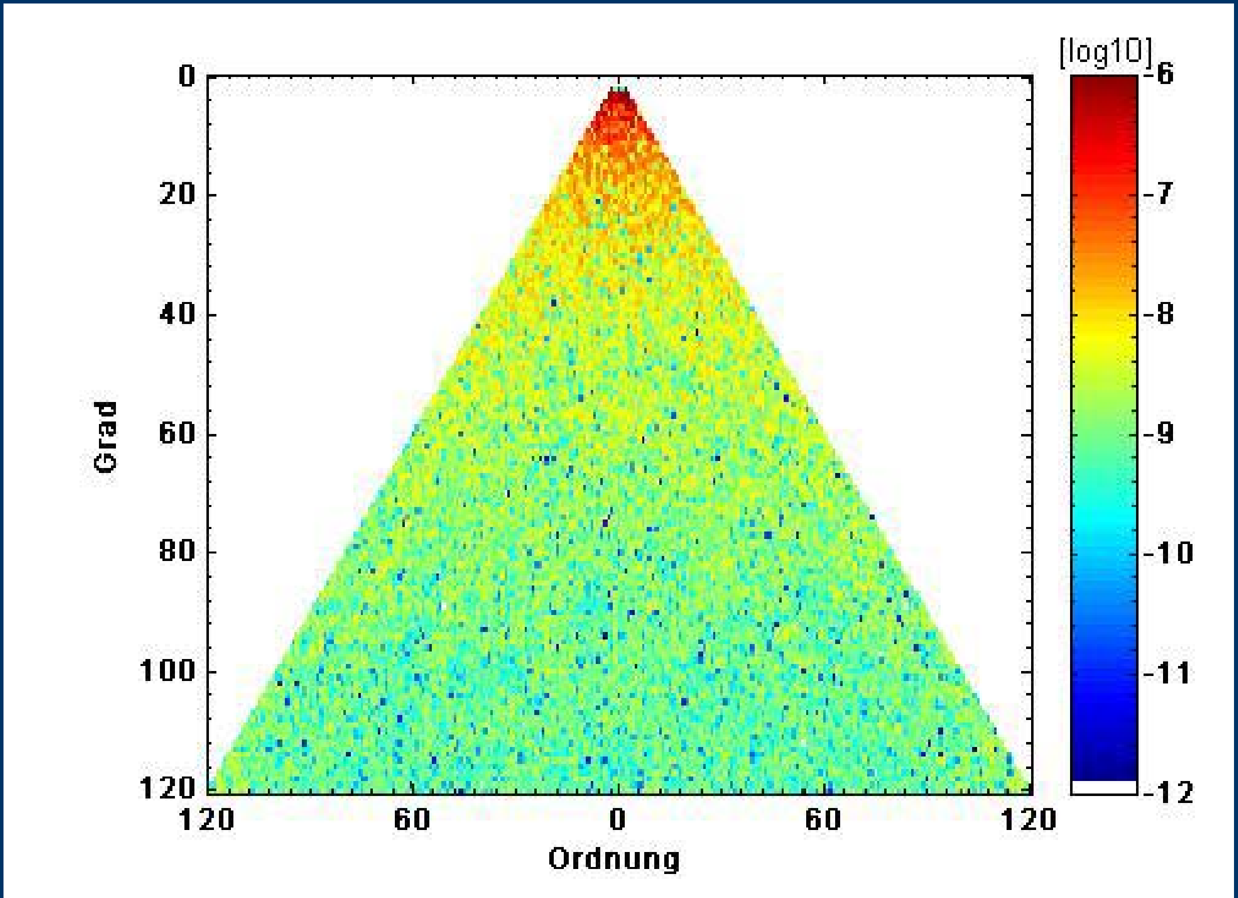
$$\delta V_A(\varphi, \lambda, r) = V_0 \sum_{n=0}^{\infty} \left(\frac{R}{r}\right)^{n+1} \sum_{m=0}^{n} \bar{P}_{nm}(\varphi)\left[\bar{C}_{nm} \cos m\lambda + \bar{S}_{nm} \sin m\lambda\right]$$

$$= V_0 \sum_{m=0}^{\infty} \left(\left[\sum_{n=m}^{\infty} \left(\frac{R}{r}\right)^{n+1} \bar{C}_{nm} \bar{P}_{nm}(\varphi) \right] \cos m\lambda + \left[\sum_{n=m}^{\infty} \left(\frac{R}{r}\right)^{n+1} \bar{S}_{nm} \bar{P}_{nm}(\varphi) \right] \sin m\lambda \right)$$

…almost a Fourier series

series representation of gravitational field

500km
333km
250km
200km



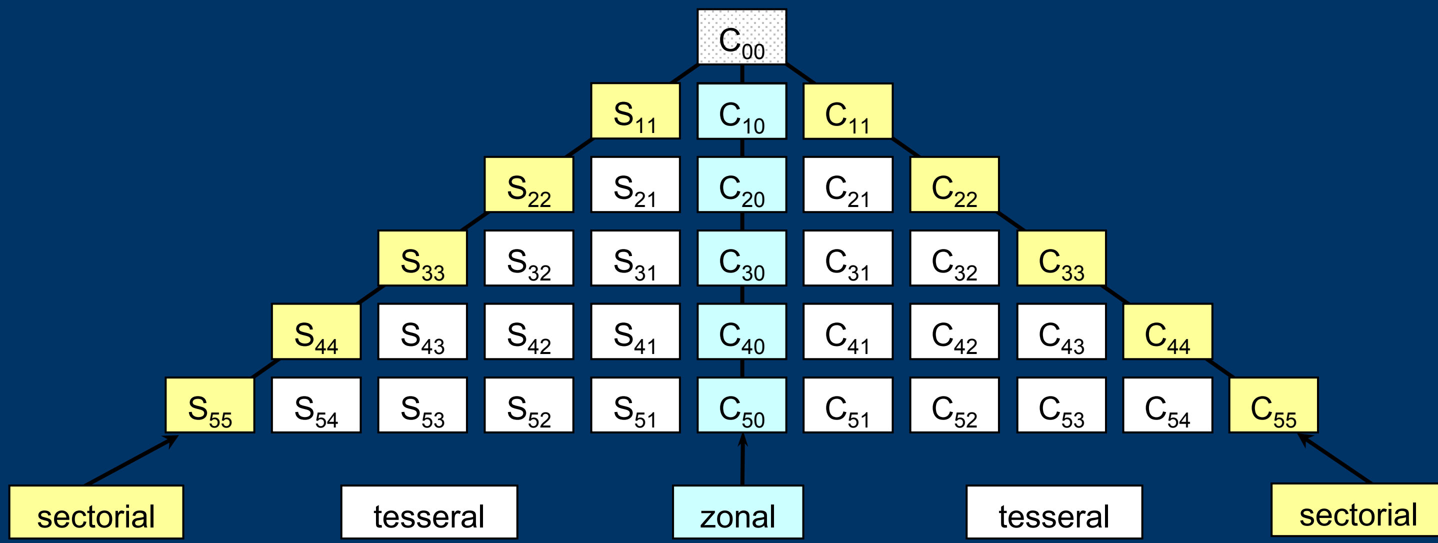
log

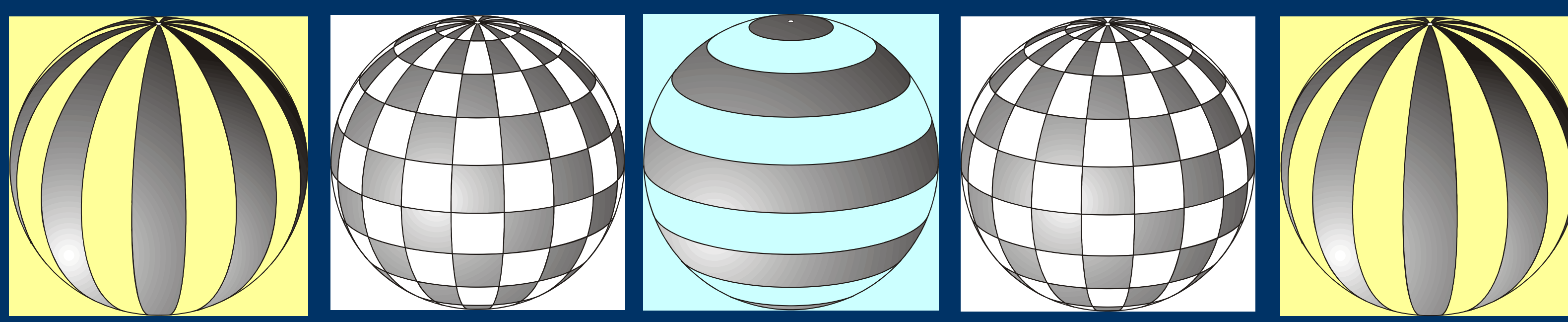
$$\lambda[km] = \frac{20\,000 km}{n \max}$$

spectral representation of the earth‘s gravitational field:

triangular plot of the spherical harmonic coefficients $\bar{C}_{nm} ; \bar{S}_{nm}$

series representation of gravitational field

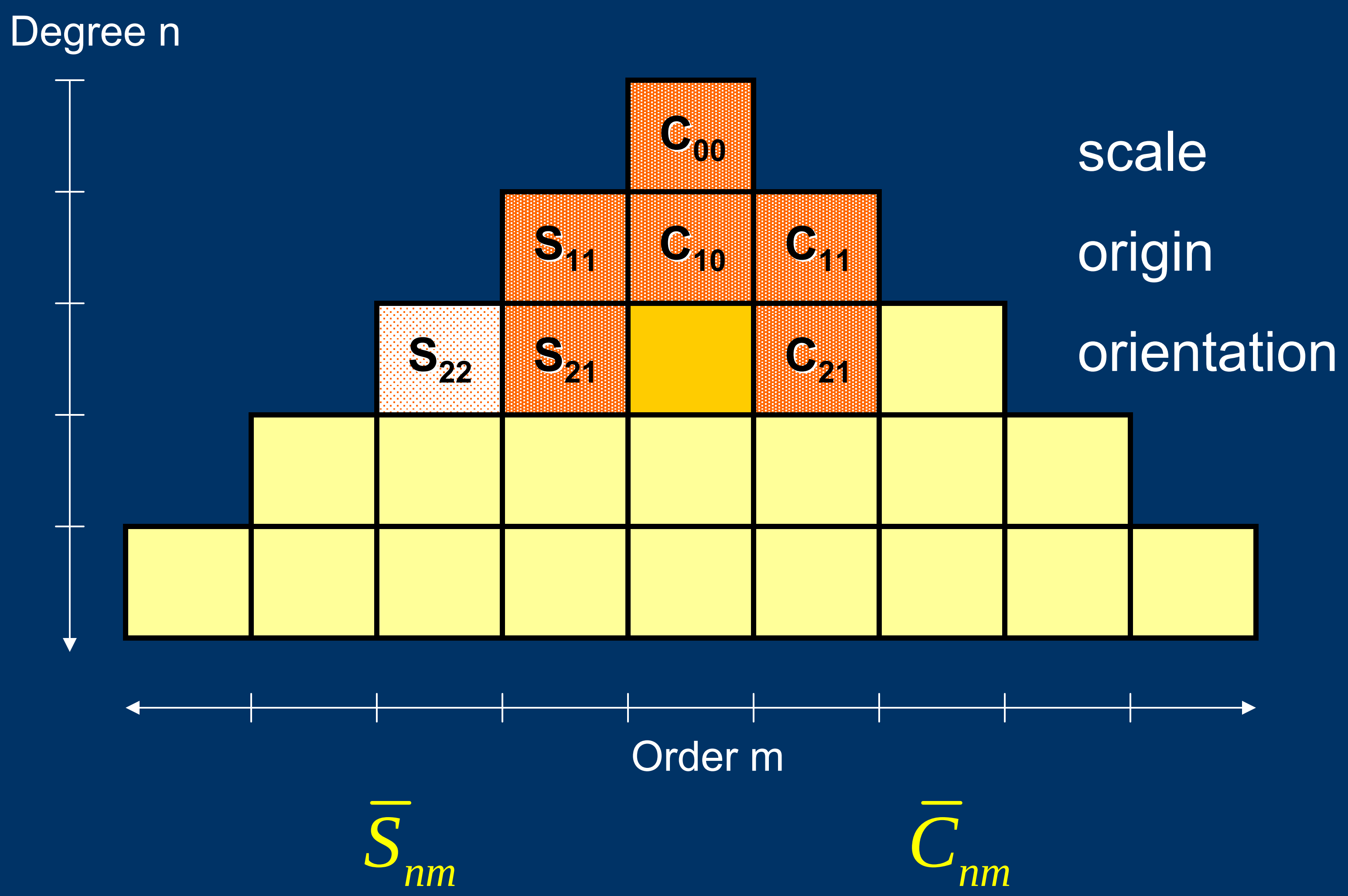




surface spherical harmonic functions:

$$Y_{nm}(\varphi,\lambda) = \bar{P}_{nm}(\varphi)\begin{cases}\cos m\lambda \\ \sin m\lambda\end{cases}$$

series representation of gravitational field

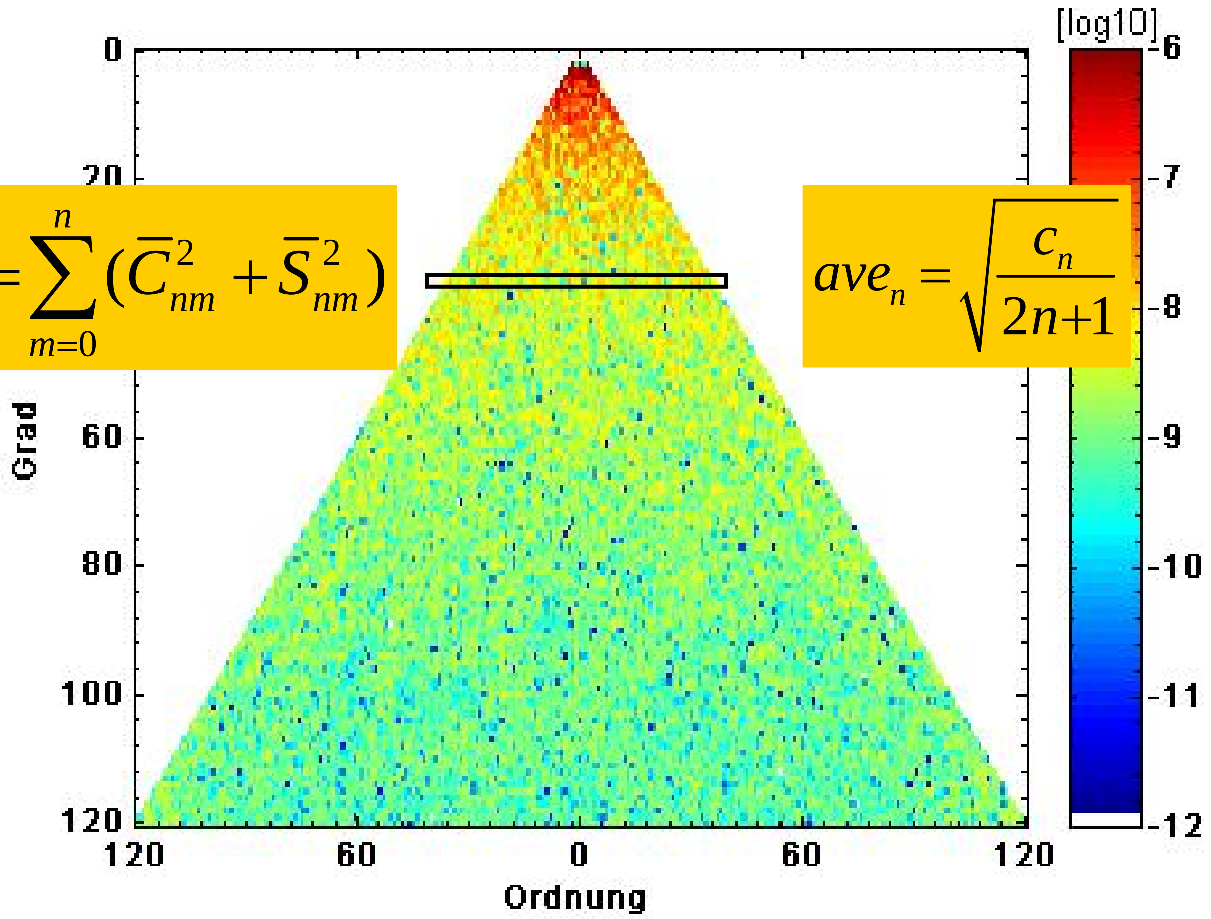


series representation of gravitational field

signal degree variance

$$c_n = \sum_{m=0}^{n} \left(\bar{C}_{nm}^2 + \bar{S}_{nm}^2 \right)$$

$$ave_n = \sqrt{\frac{c_n}{2n+1}}$$

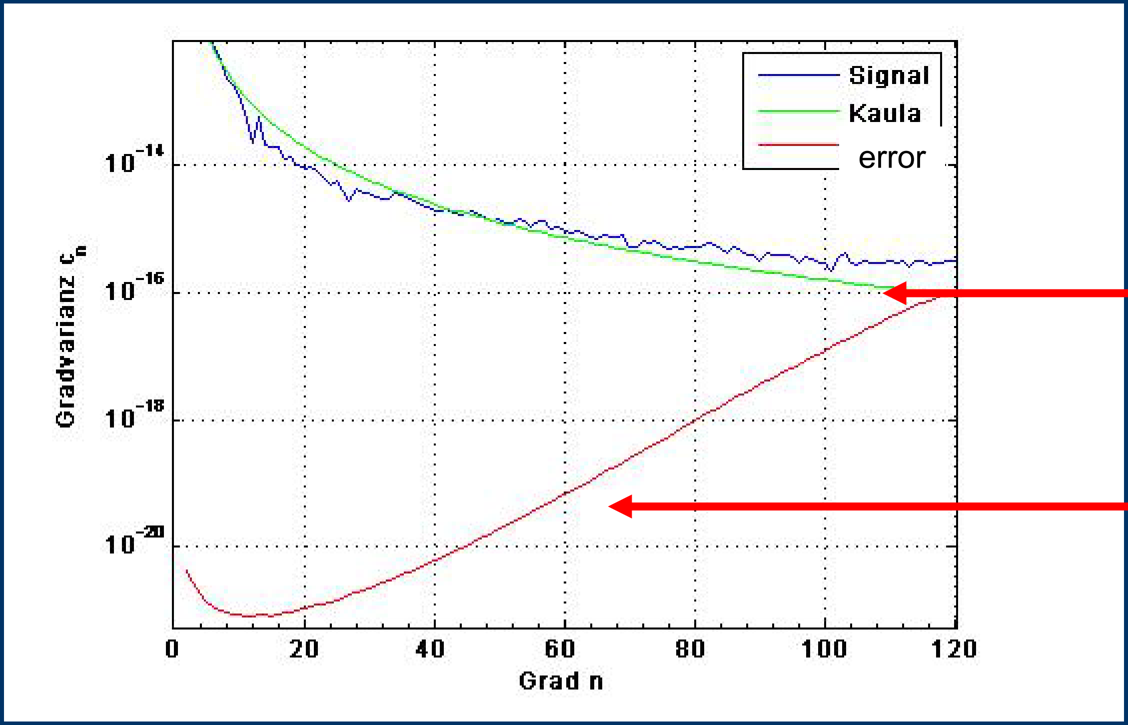


in analogy to signal processing: characteristics of signal and noise

Here: degree variances correspond to power spectral density

series representation of gravitational field

power „law“ by WM Kaula

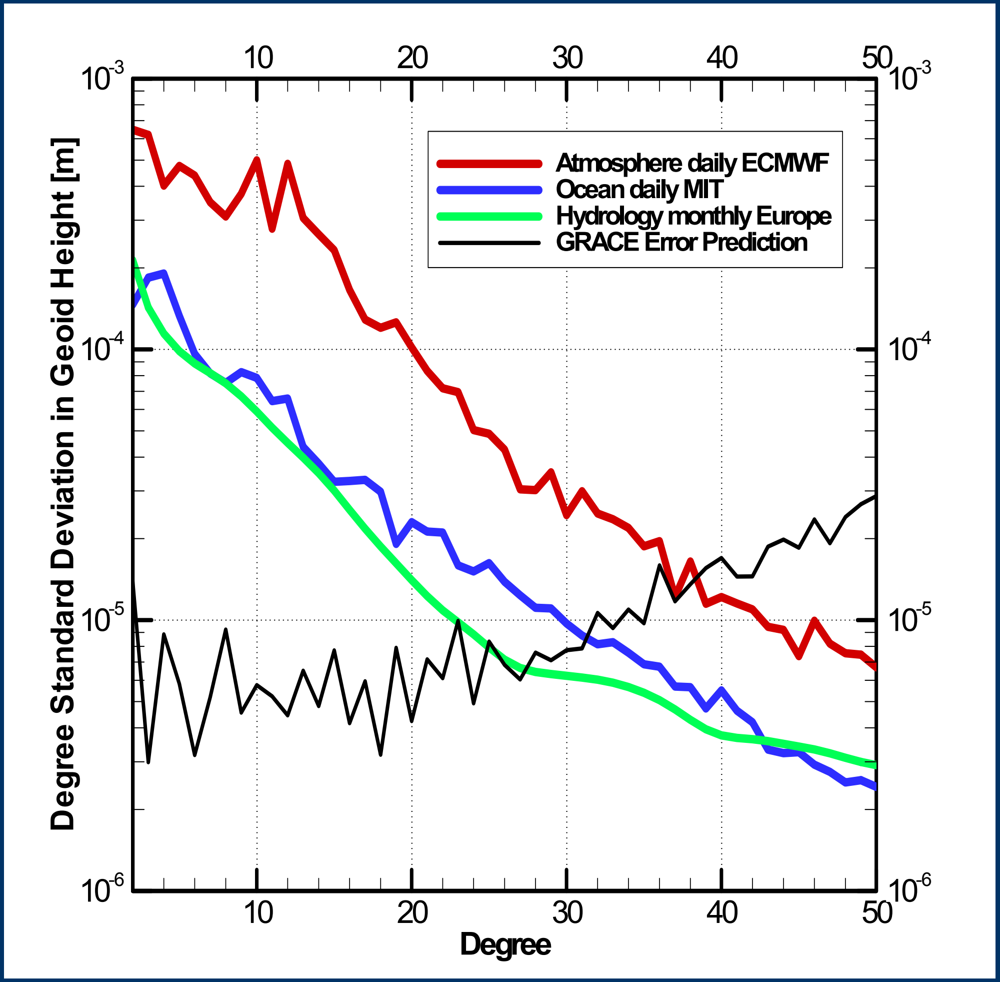


$$c_n = 1.6 \cdot 10^{-10} / n^3$$

$$\sigma_n^2$$

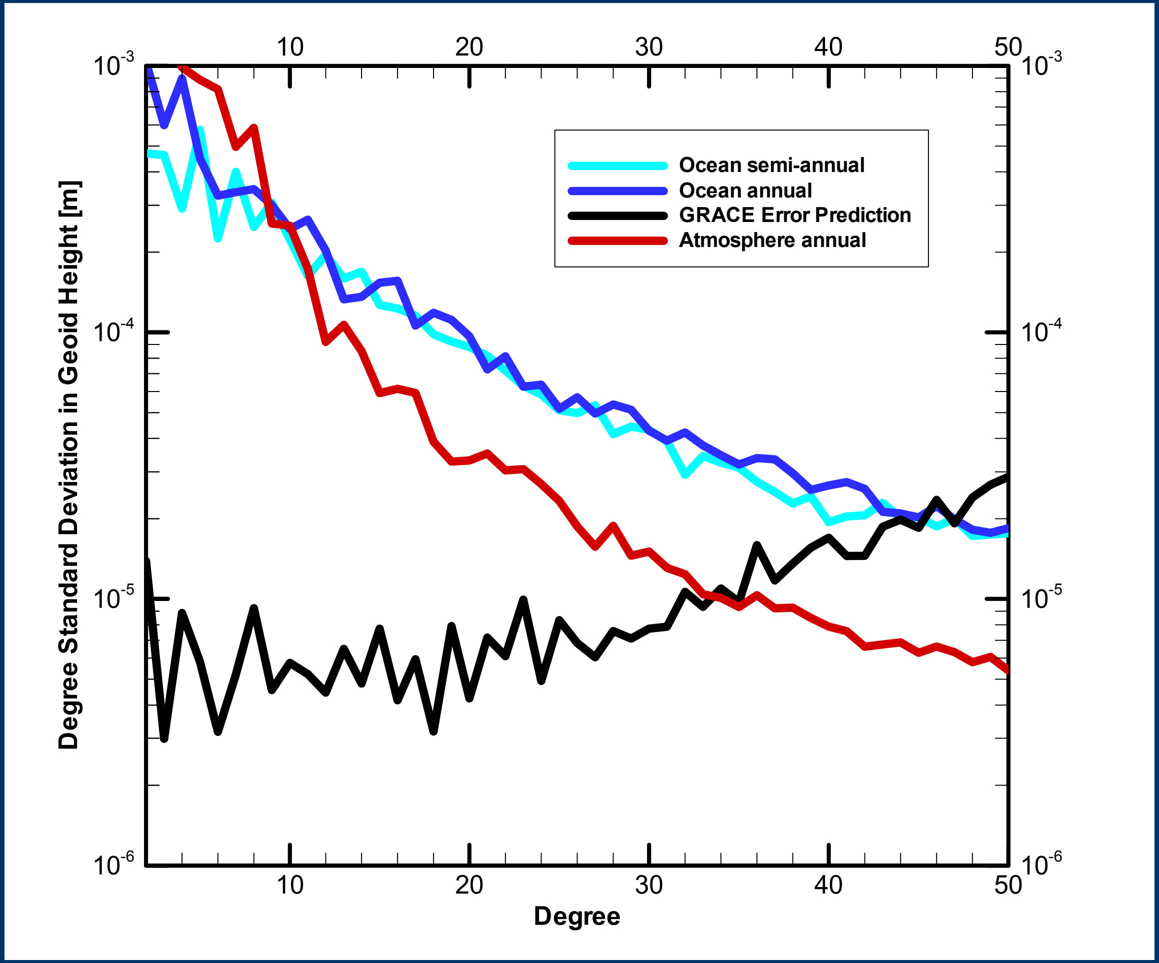
signal and error degree variances (dimensionless)

series representation of temporal variations of gravitational field



rapid time variable geoid signals (RMS)

[to be divided by the earth radius in order to arrive at dimensionless units] series representation of temporal variations of gravitational field



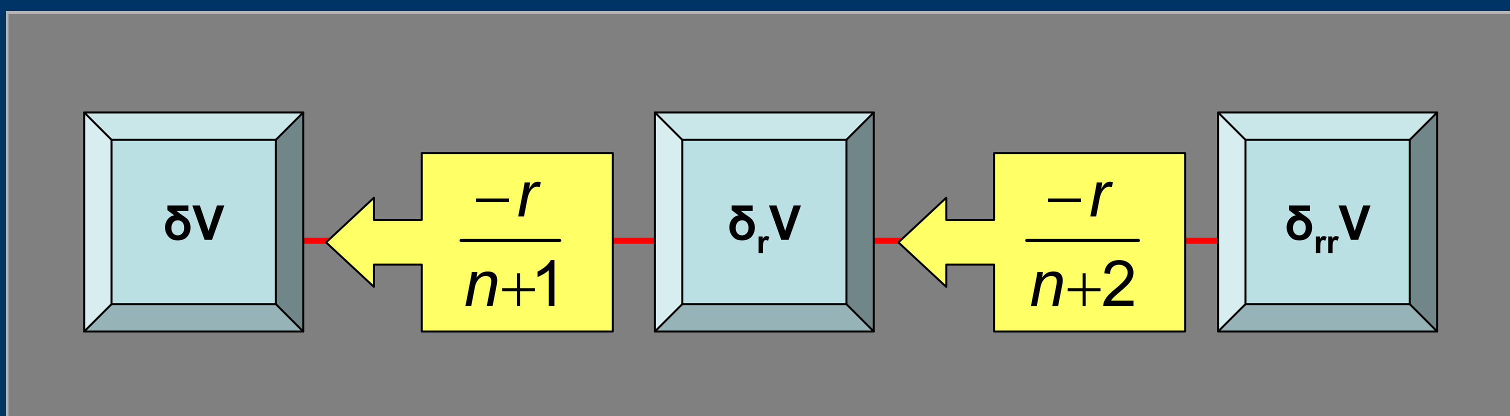
“slow” geoid time variable signals (RMS)

[to be divided by the earth radius in order to arrive at dimensionless units]

three levels of gravity quantities on earth and in space

$$\delta V_A(\varphi,\lambda,r) = V_0 \sum_{n=0}^{\infty}\left(\frac{R}{r}\right)^{n+1}\sum_{m=0}^{n}\bar{P}_{nm}(\varphi)\left[\bar{C}_{nm}\cos m\lambda + \bar{S}_{nm}\sin m\lambda\right]$$

$$= V_0 \sum_{m=0}^{\infty}\left(\left[\sum_{n=m}^{\infty}\left(\frac{R}{r}\right)^{n+1}\bar{C}_{nm}\bar{P}_{nm}(\varphi)\right]\cos m\lambda + \left[\sum_{n=m}^{\infty}\left(\frac{R}{r}\right)^{n+1}\bar{S}_{nm}\bar{P}_{nm}(\varphi)\right]\sin m\lambda\right)$$

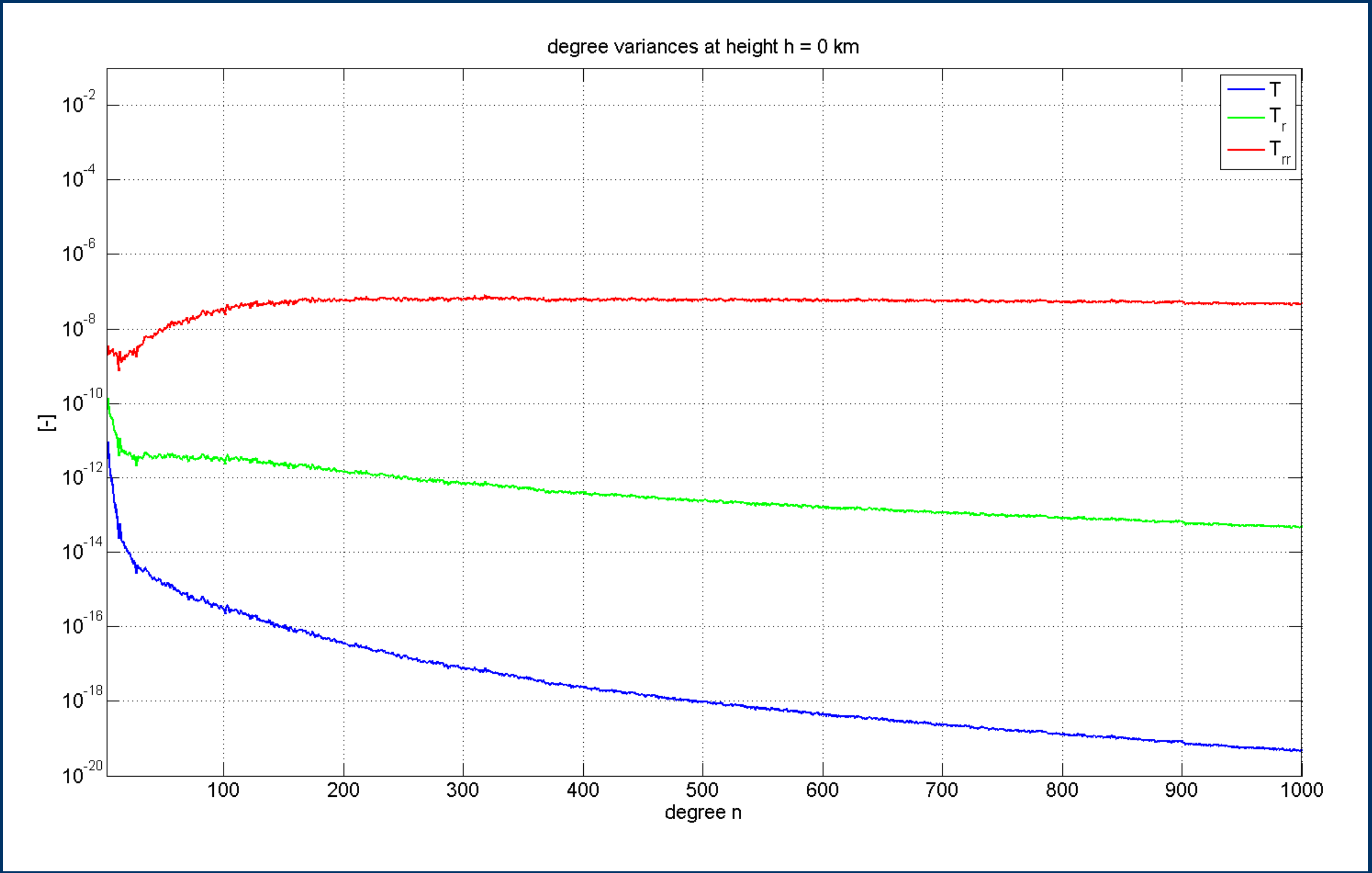


disturbance potential or geoid

gravity disturbances or gravity anomalies

gravity gradients or torsion balance

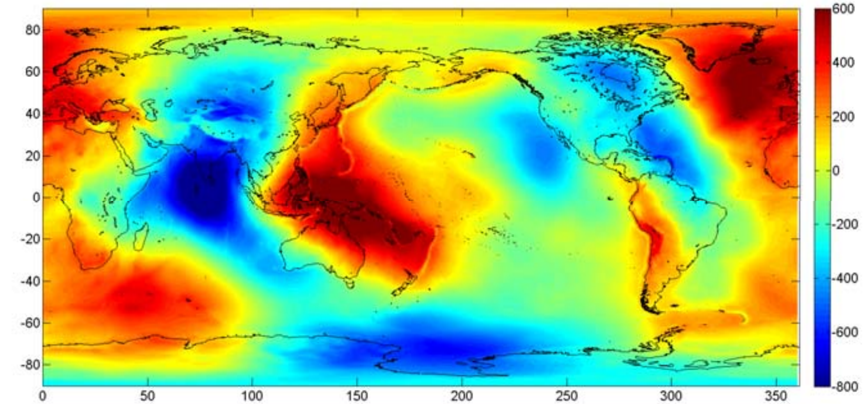
various gravity quantities on earth and in space



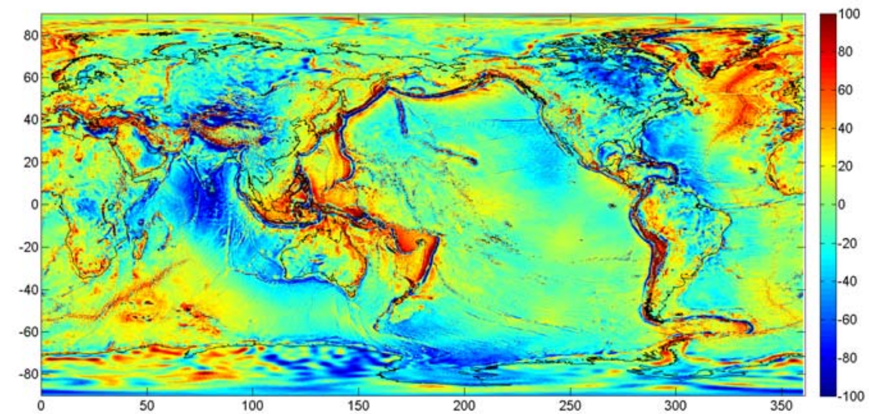
Gravity model EGM08, D/O1000

h = 0km

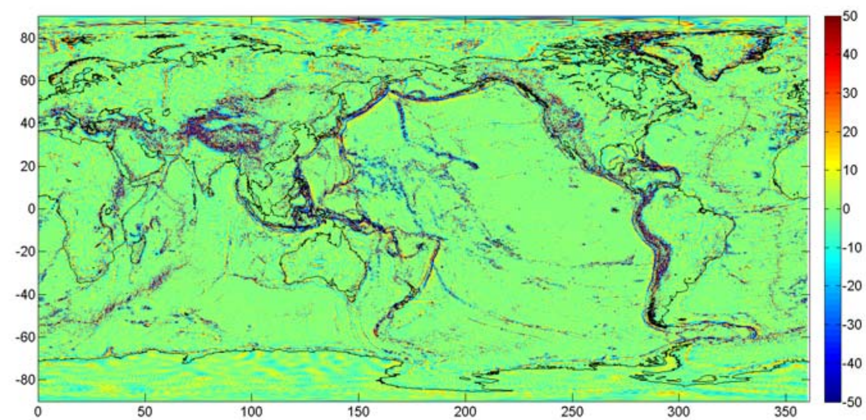
δV



δV_r



δV_{rr}

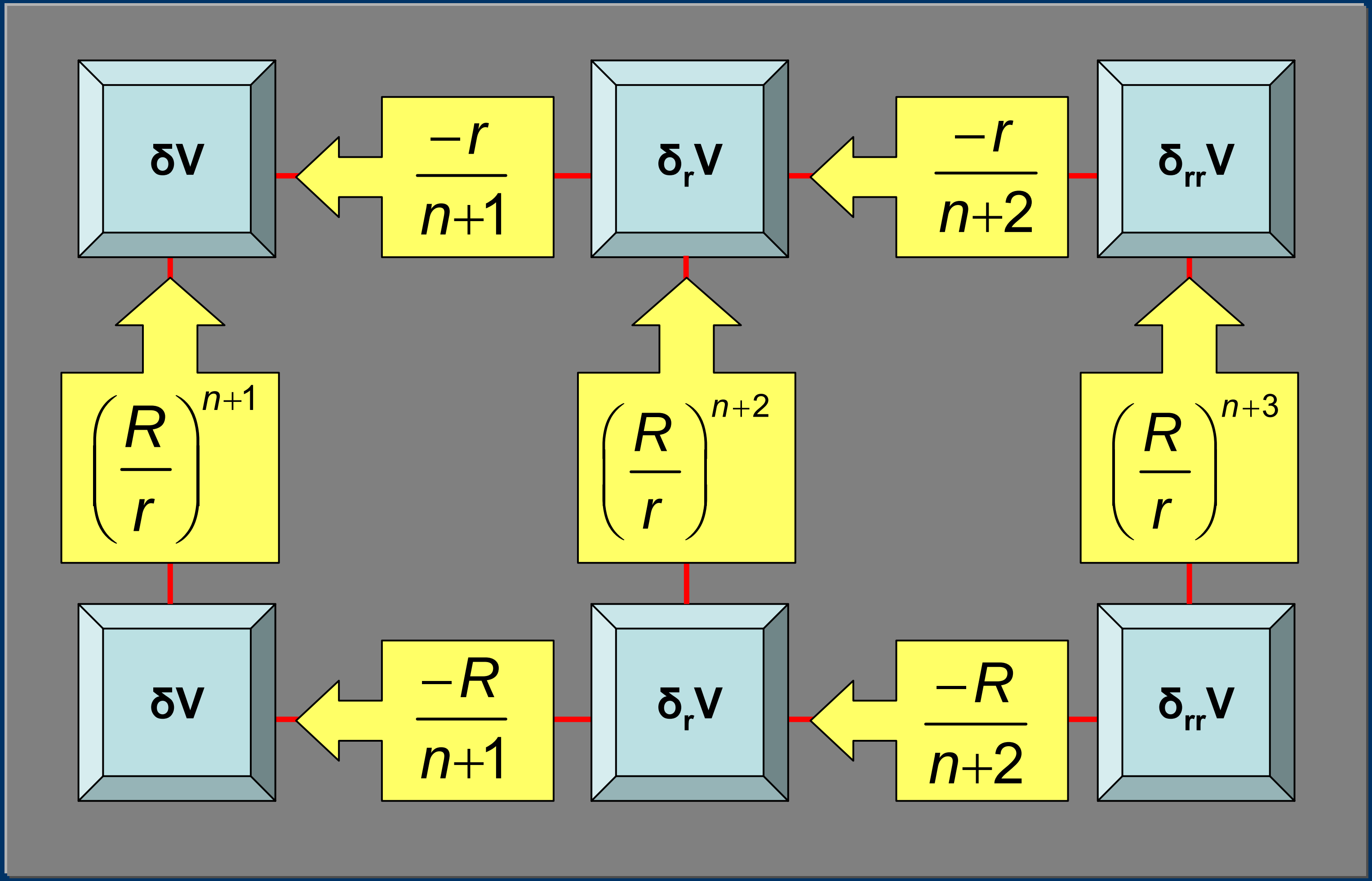


Gravity model EGM08, D/O1000

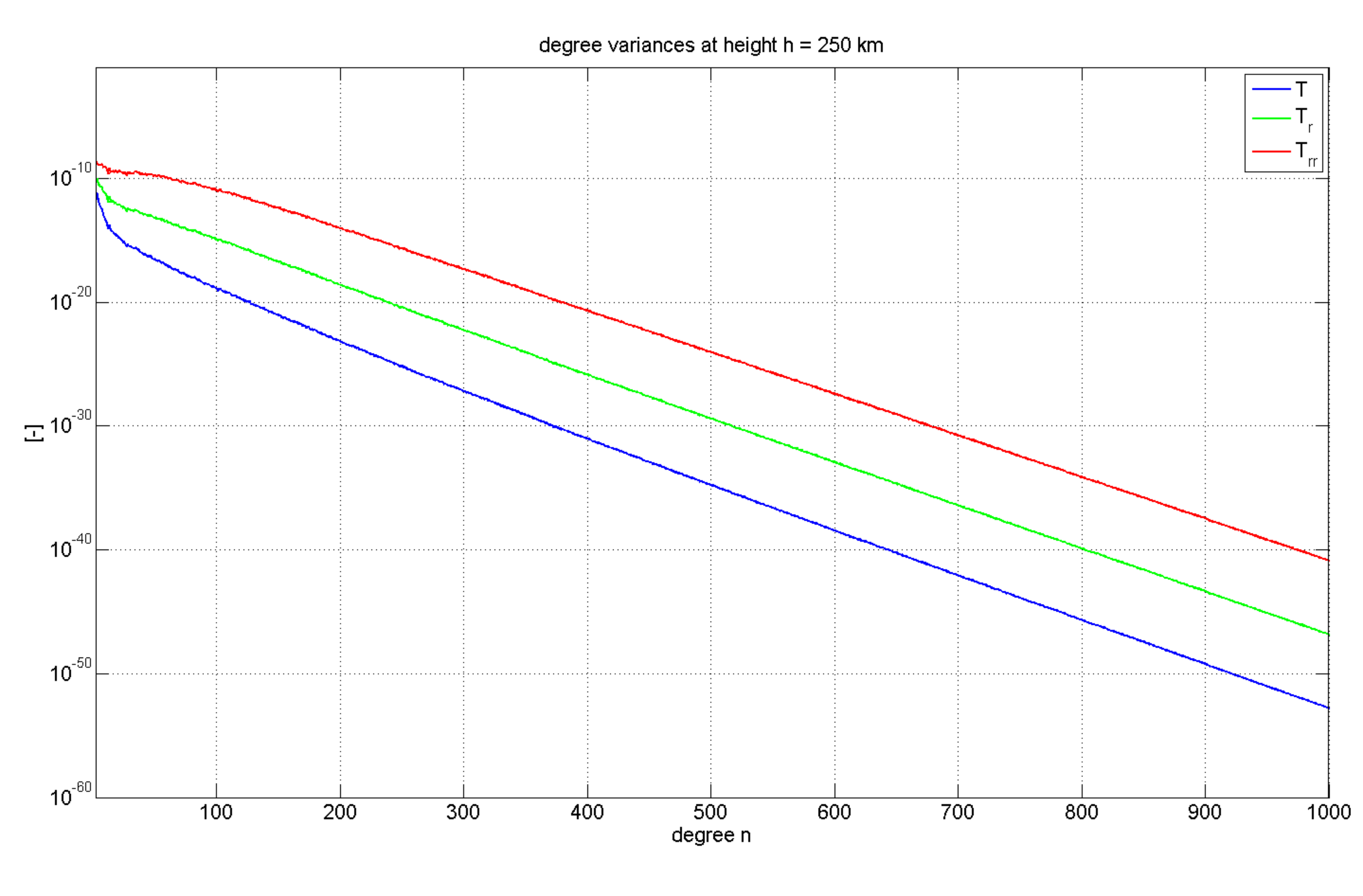
three levels of gravity quantities on earth and in space

satellite altitude: r

earth's Surface: R



various gravity quantities on earth and in space



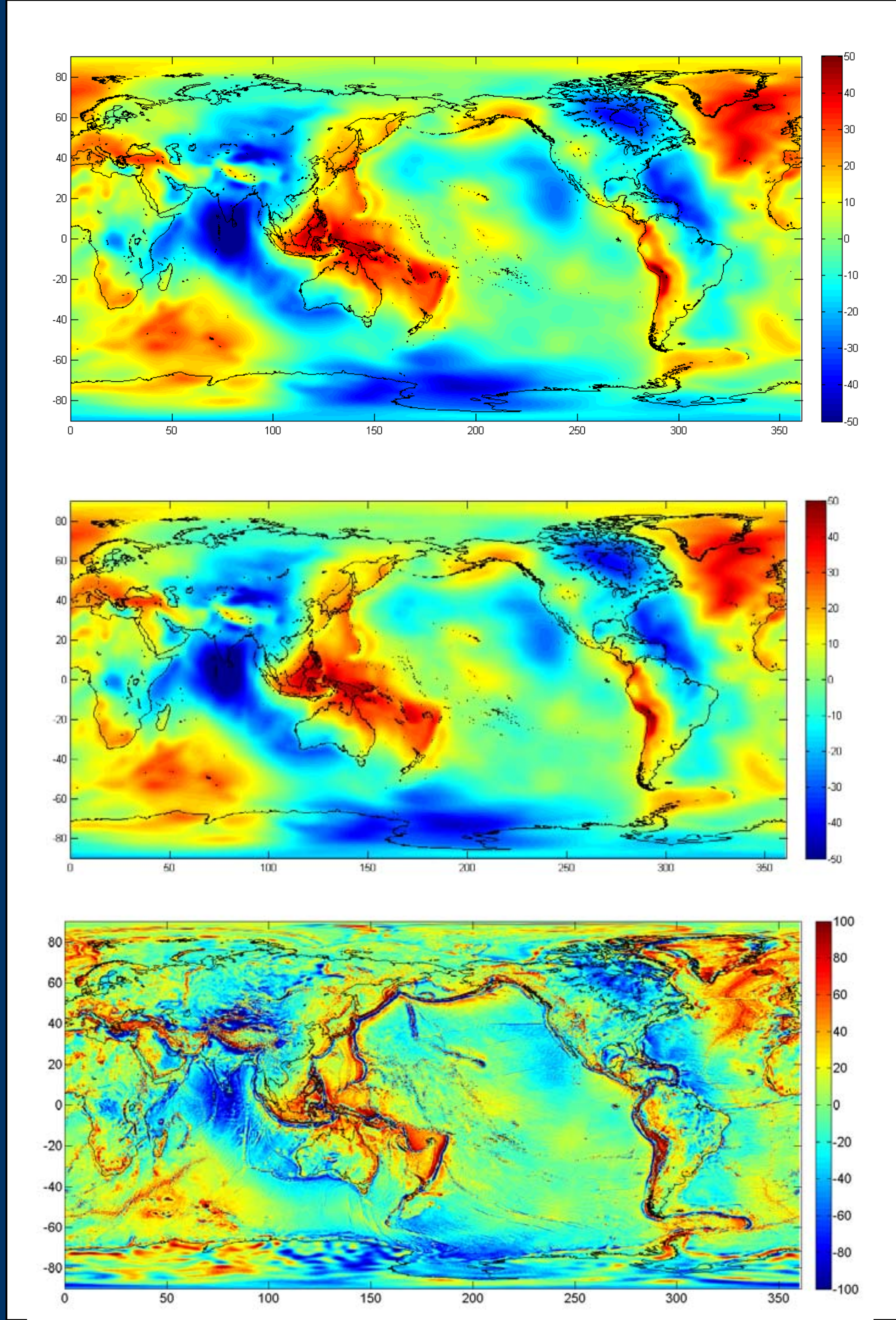
Gravity model EGM08, D/O1000

δV_r

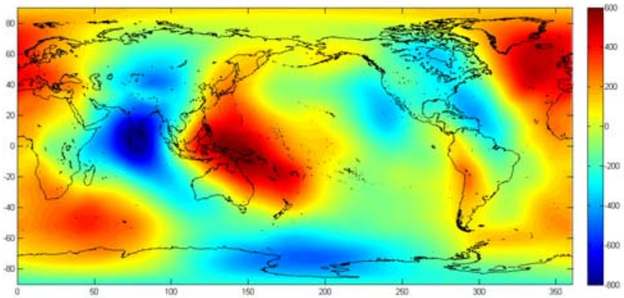
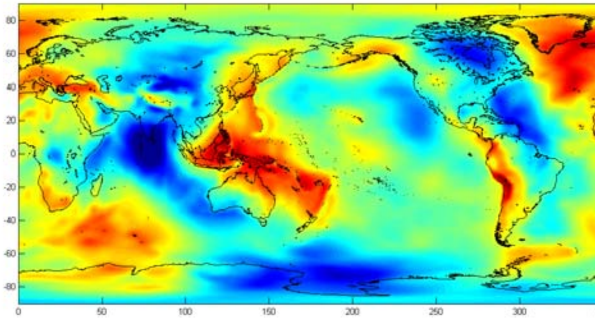
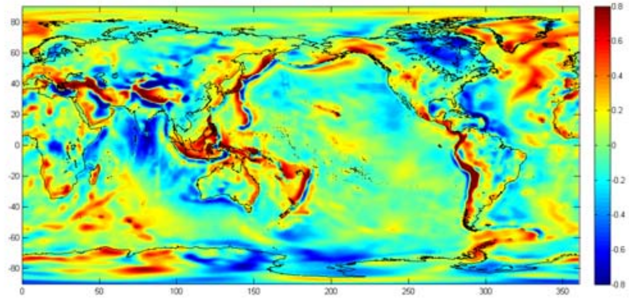
h = 400km

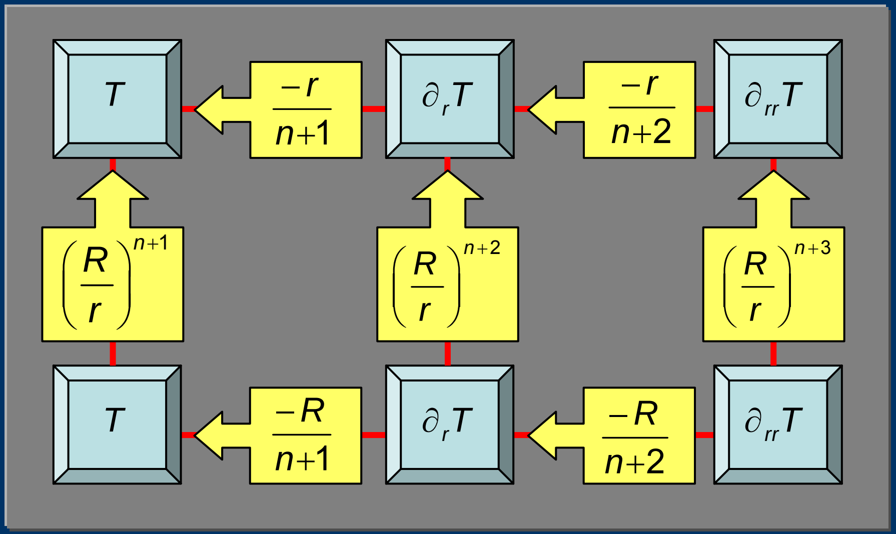
h = 250km

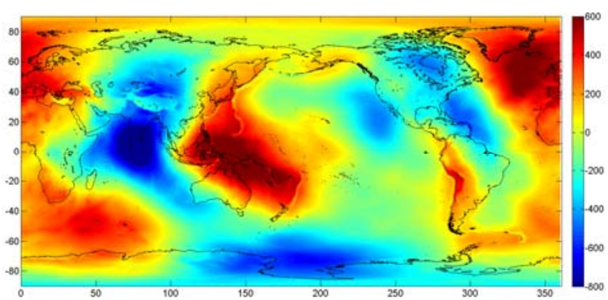
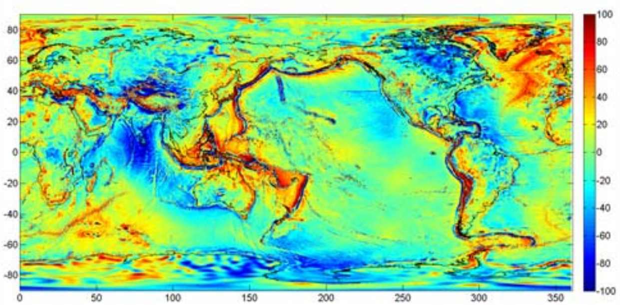
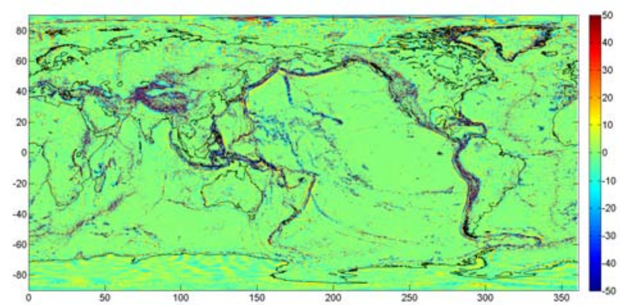
h = 0km



Gravity model EGM08, D/O1000

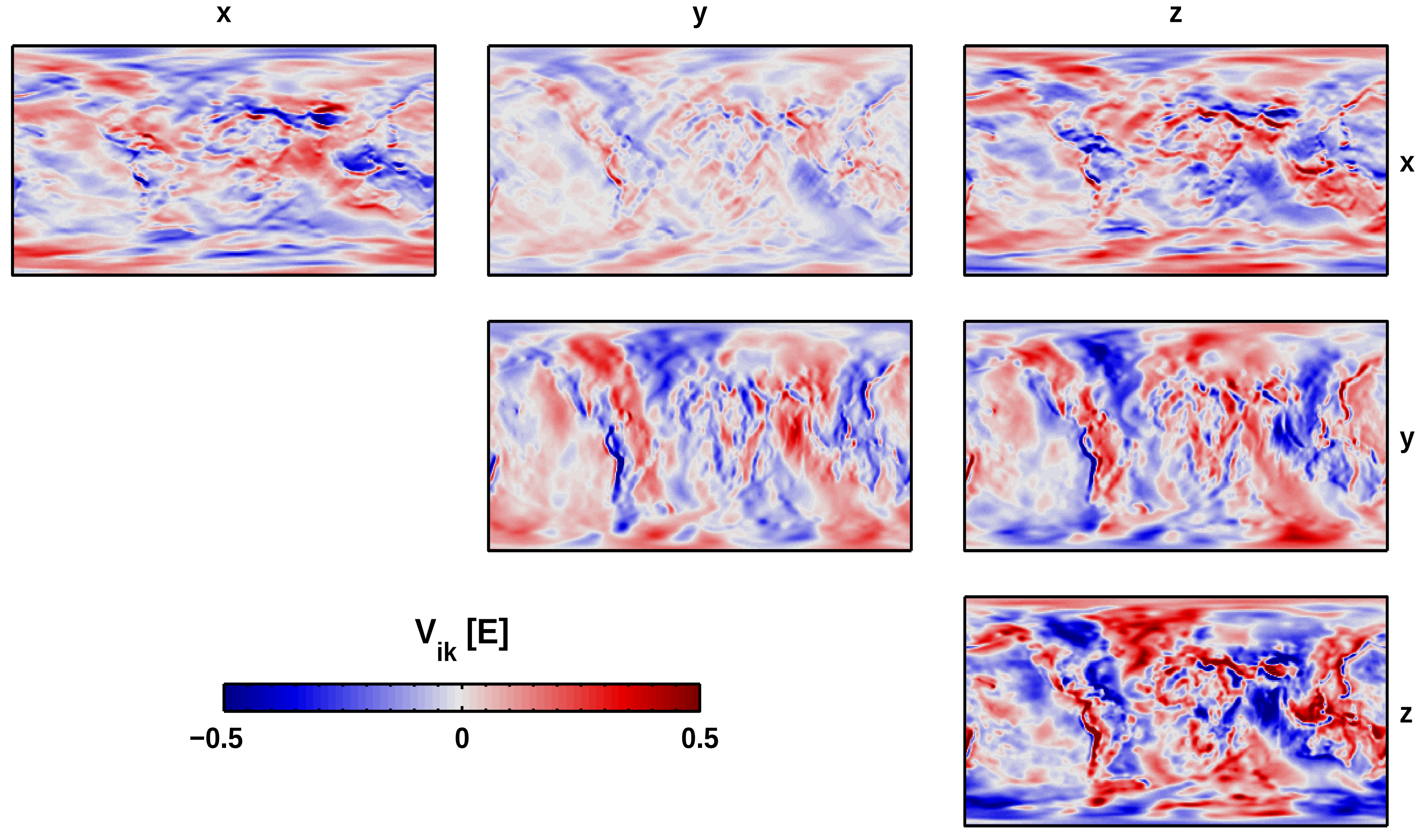






Gravity model EGM08, D/O1000

various gravity quantities on earth and in space



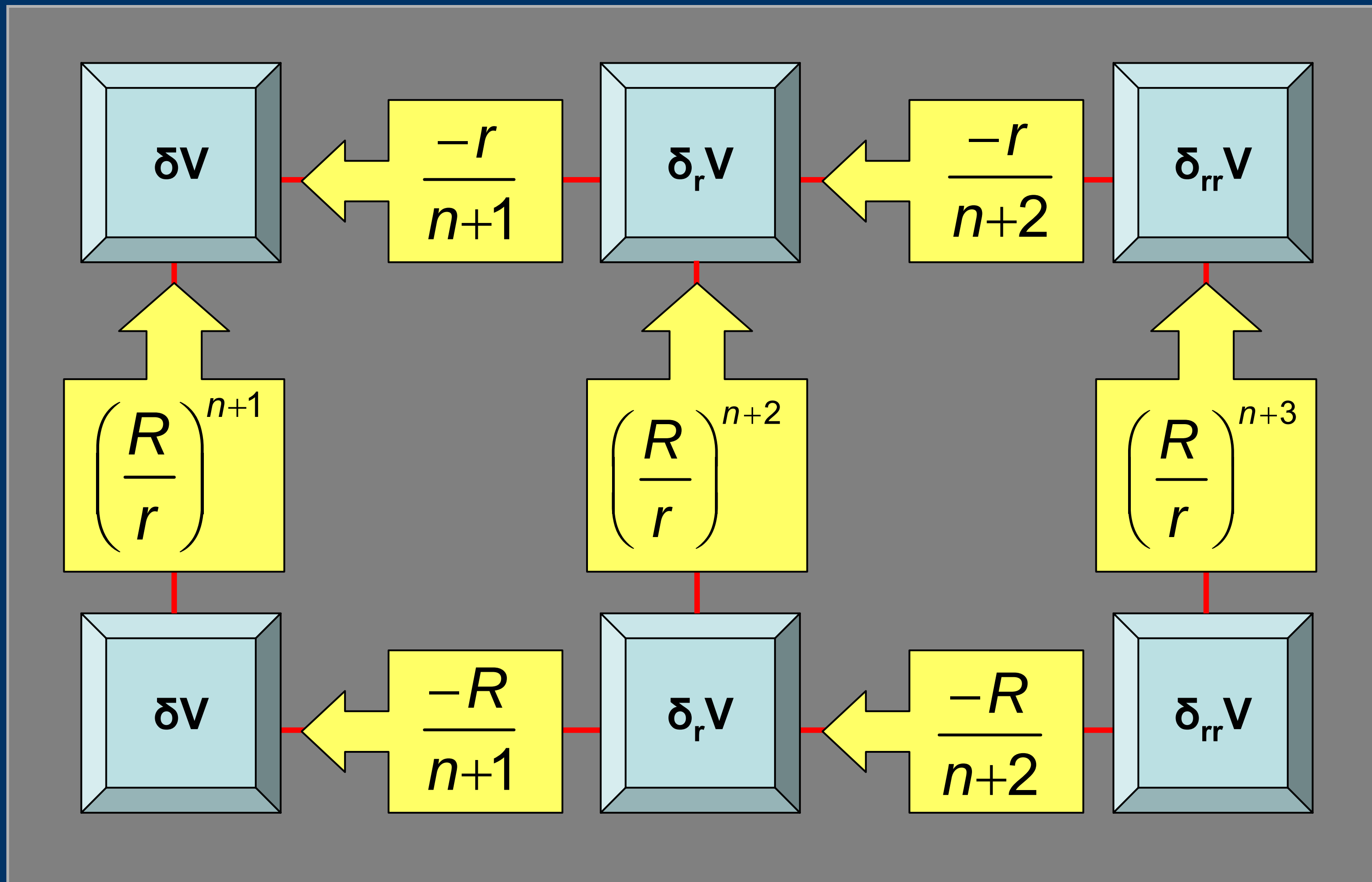
satellite-to-satellite tracking low-low

satellite-to-satellite tracking high-low

satellite gradiometry

satellite altitude r

earth‘s surface R



disturbance potential or geoid

gravity disturbances or gravity anomalies

gravity gradients or torsion balance

summary of lecture One

1. Newton‘s law of gravitation describes all its relevant properties such as inverse square distance, principle of superposition, its stationary part being vorticity free, and source free outside the earth (Laplace equation)
2. Gravity is the sum of gravitation and the centrifugal part
3. Satellite orbits are essentially described by gravitation
4. Tides are an accelertation (a force) relative to the earth‘s center of mass
5. The global gravitational field is represented as a series of spherical harmonics being a solution of Laplace partial differential equation (*Dirichlet*)
6. Spherical harmonics on a sphere are analogous to a Fourier series in a plane
7. Therefore there exists a closed theory of „signal and noise processing“
8. With increasing distance from the earth sphere the series coefficients are dampening out per degree n like $(R/R+h)^{n+1}$
9. With each radial derivative of the gravitational potential the series coefficients are amplified per degree n like $(n+1)$
10. The strategy of satellite missions GRACE and GOCE rests on the principle of compensating the dampering effect by amplification (see 8. and 9.)